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# "Eigen-periodic"-in-space surface heating in conduction with application to conductivity measurement of thin films

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ABSTRACT

A two-dimensional heat conduction problem in Cartesian coordinates subject to a periodic-in-space boundary condition is analyzed by the Green's functions approach. It is pointed out that when the frequency of the spatial periodic heating equates one of the natural frequencies (eigenvalues) of the system, the solution of the 2D heat conduction problem can be written down very simply as the product of the periodic surface condition (termed the "eigen-periodic") by the solution of a 1D fin problem along the nonhomogeneous direction. This result suggests a novel and simple algebraic equation for determining the thermal conductivity of thin films placed on substrates under steady state conditions. High space frequencies of the sinusoidal heating, larger than the deviation frequency, are used to make negligible the thermal deviation effects due to the presence of the substrate.



## 1. Introduction

The exact analytical study of periodic-in-space boundary conditions in transient and steady state heat conduction is relevant for various reasons. One is for verification purposes of large multi-dimensional numerical heat transfer codes [1–4] and related intrinsic verification of exact analytical solutions [5]. Another is for determining the thermal conductivity of thin films [6–11] (used in a variety of micro-electromechanical systems, i.e., MEMS) placed on substrates under steady state conditions. (In this case, the spatially periodic heating can experimentally be obtained by using the pulsed-laser interference fringes as described in Refs. [7–9].) Also, contrary to what happens in thermal convection, where spatially periodic boundary temperatures in porous media were investigated [12–14], in heat conduction spatial periodic heating (or cooling) is somewhat neglected.

In this paper, we consider a 2D transient heat conduction problem subject to a periodic-in-space surface heat flux which satisfies the boundary conditions in the homogeneous direction. Also, the body is initially at zero temperature and the remaining boundary conditions of the 1st kind are homogeneous (Section 2). The surface condition (termed the "eigen"-periodic) of the present study is motivated by the fact that it reduces the dimensions of the problem (in the current case, 2D → 1D). In this case, no thermal deviation effects are caused by the homogeneous boundary conditions perpendicular to the heating surface. Therefore, the solution of the problem obtained by Green's functions (Section 3) can simply be written as the product of the "eigen-periodic" surface condition by the solution of a 1D fin transient problem along the nonhomogeneous direction of the 2D slab (Section 4).

Different boundary conditions at the face parallel to the surface with an "eigen-periodic"-in-space heat flux are then investigated (Section 5). Special cases including the boundary condition of the zero-th kind (semi-infinite solid) and the perfect thermal contact of the first body to a semi-infinite body are also analyzed.

To derive a simple algebraic equation that involves a property of engineering interest such as the thermal conductivity, we analyze the thermal penetration effects due to an "eigen-periodic"-in-space heating; also analyzed are the thermal deviation effects due to a homogeneous boundary condition (where this boundary can be parallel or perpendicular to the surface heated by a spatial "eigen-periodic" heat flux). For that purpose, we define penetration and deviation frequencies analogous to the definition of penetration and deviation times given by the same authors in [15].

The penetration frequency is defined as the frequency that is needed for the steady state temperature at an interior point in a 2D semi-infinite solid to be just affected by an "eigen-periodic"-in-space heating at a boundary surface (Section 6). The deviation

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**Nomenclature**

| | |
|---|---|
| $C_f$ | dimensionless constant related to the dimensionless frequency $\beta_f$ of the periodic-in-space heating, $\beta_f W/L$ |
| $G$ | Green's function (subscript designates the boundary conditions) ($m^{-1}$) |
| $k$ | thermal conductivity (W/(m K)) |
| $L$ | rectangle length in the $x$-direction (m) |
| $q_0$ | heat flux ($W/m^2$) |
| $t$ | time (s) |
| $T$ | temperature (K) |
| $T_y$ | temperature along $y$ (K) |
| $u$ | 'cotime', $t-\tau$ (s) |
| $W$ | rectangle length in the $y$-direction (m) |
| $x$, $y$ | space coordinates (m) |
| $X$, $Y$ | eigenfunctions along $x$, $y$ |
| $\alpha$ | thermal diffusivity ($m^2/s$) |
| $\beta_f$ | dimensionless frequency of the periodic-in-space heating, $\nu_f L$ |
| $\beta_m$ | $m$ th dimensionless eigenvalue in the $x$-direction, $\nu_m L$ |
| $\eta_n$ | $n$ th dimensionless eigenvalue in the $y$-direction, $\gamma_n W$ |
| $\gamma_n$ | $n$ th eigenvalue along $y$ ($m^{-1}$) |
| $\nu_f$ | frequency of the periodic-in-space heating ($m^{-1}$) |
| $\nu_m$ | $m$ th eigenvalue along $x$ ($m^{-1}$) |
| $\theta$ | dimensionless temperature along $y$, $T_y/T_K$ |

*Subscripts*

| | |
|---|---|
| *c.t.* | complementary transient component |
| $F$ | fin |
| *I, J* | indices, i.e., 1, 2 or 3. These integers denote boundary conditions of 1st, 2nd or 3rd kind at the faces $x = 0$ and $x = L$ of the rectangle |
| *K, L* | indices, i.e., 1, 2 or 3. These integers denote boundary conditions of 1st, 2nd or 3rd kind at the faces $y = 0$ and $y = W$ of the rectangle |
| *s.s.* | steady-state component |
| $x$, $y$ | in the $x$- and $y$-direction |

*Superscript*

| | |
|---|---|
| $\sim$ | dimensionless (space: $\tilde{x}=x/L$ and $\tilde{y}=y/W$; time $\tilde{t}=\alpha t/W^2$ and $\tilde{u}=\alpha u/W^2$; Green's function: $\tilde{G}_{XIJ}=G_{XIJ}L$ and $\tilde{G}_{YKL}=G_{YKL}W$) |

frequency is however defined as the frequency that is needed for the steady state temperature at an interior point of a 2D finite solid (heated at a boundary through an "eigen-periodic"-in-space source) to be just affected by the presence of a homogeneous boundary condition (Section 7). By "just affected" we mean to some sufficiently small numerical value such as $10^{-2}$ (typical of thermal boundary layer thickness) or even the much smaller value of $10^{-10}$.

Once the penetration and deviation frequencies are calculated, a simple algebraic equation involving thermal conductivity can be found (Section 8). In the thermal grating technique [7–9] used for thin films, the pulsed-laser excitation takes a short while so allowing measurements of the thermal diffusivity by monitoring the temperature changes with time. In contrast, in proposed technique the pulsed-laser excitation is applied until the steady state condition is reached. As a matter of fact, the solution does have a steady state but the implementation experimentally has a quasi-steady state. We need to add a constant heat flux component so that the heat flux never becomes negative (Section 8). For that reason, the proposed solution can fall among the so-called steady state (film-on-substrate) techniques ([6], Section 3.4.2). It has the advantage that a temperature gradient can be formed in the direction essentially perpendicular to that of heat flux, so allowing measurements of the thermal conductivity of thin layers.

## 2. Problem formulation

A two-dimensional transient heat conduction problem with a periodic-in-$x$ heat flux variation applied to the $y = 0$ face of the rectangle $L$ by $W$ is shown in Fig. 1. The boundary conditions at $x = 0$, $x = L$ and $y = W$ are homogeneous and of the 1st kind. Also, the temperature is initially uniform which without loss of generality can be zero and the thermal properties are independent of location and temperature.

The above problem can be denoted by *X*11*B*00 *Y*21*B*(*x*6)0*T*0. (See [[16], Chapter 2] for a fuller discussion of the numbering system devised by Beck et al. for Cartesian heat conduction.) In brief, $X$ and $Y$ denote the Cartesian directions; "($x$6)" denotes a sinusoidal space function in the $x$-direction; and *T*0 indicates a zero initial temperature.

Now, the periodic variation along $x$ of the heat flux applied at $y = 0$ is taken as the solution of the eigenvalue problem in the $x$-direction, namely $X_f(\beta_f, x/L) = \sin(\beta_f x/L)$, where $\beta_f = f\pi$ with $f = 1, 2, 3, \ldots$ For that reason, it will be termed the "eigen-periodic" boundary condition and will use "($xE$)" as a notation afterwards.

A mathematical statement of the above linear heat conduction problem is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (0<x<L,\ 0<y<W,\ t>0) \tag{1a}$$

$$T(0,y,t)=0 \quad T(L,y,t)=0 \tag{1b}$$

$$-k\left(\frac{\partial T}{\partial y}\right)_{y=0}=q_0\sin\left(\beta_f\frac{x}{L}\right) \quad T(x,W,t)=0 \tag{1c}$$

$$T(x,y,0)=0 \tag{1d}$$

where the only nonhomogeneous term (or driving term) is in Eq. (1c).

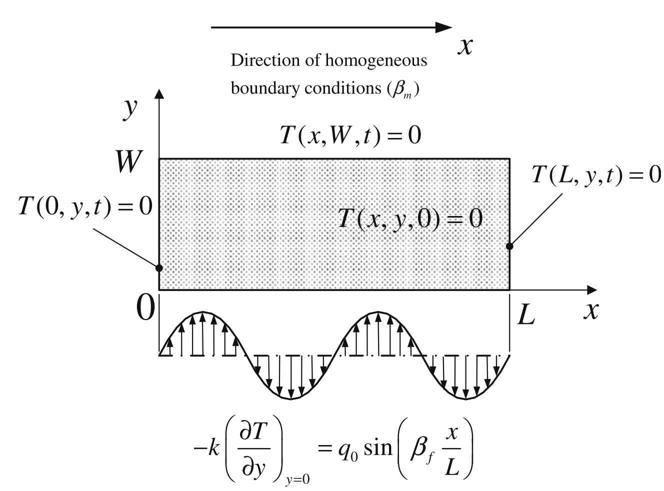


**Fig. 1.** Transient heat conduction problem in a rectangle with zero initial temperature and homogeneous boundary conditions of the first kind at all boundary surfaces except an "eigen-periodic" heat flux variation along $x$ at $y = 0$. Problem notation is *X*11*B*00 *Y*21*B*(*xE*)0*T*0.

## 3. Solution procedure

The above 2D transient heat conduction problem may be solved using conventional solution procedures such as the separation of variables (SOV) approach [17] and the Green's functions approach [16,18,19]. However, as this problem is nonhomogeneous due to the nonhomogeneity of the boundary condition at $y = 0$, the standard SOV method cannot be applied in a straightforward and efficient manner. Its application requires that the heat conduction problem (1) be split up into two simpler ones (that may then be solved using SOV) in the following manner:

1. A nonhomogeneous steady-state problem with a nonhomogeneous boundary condition at $y = 0$ (denoted $X11B00$ $Y21B(xE)0$) whose solution gives a contribution to the so-called steady-state component $T_{s.s.}(x, y) = T(x, y, \infty)$ of the complete temperature;
2. A homogeneous time-dependent problem whose initial temperature becomes the negative of the previous steady state result. Its solution gives what we call the "complementary transient" component $T_{c.t.}(x, y, t)$. (Problem notation is $X11B00$ $Y21B00T(x-y-)$.)

As regards Green's functions (GF), they are very powerful and appropriate tools for obtaining solutions of nonhomogeneous heat conduction problems, including time- and space-variable boundary conditions and time- and space-variable volume energy generation. For these reasons, the problem given by Eqs. (1a)–(1d) is solved next using Green's functions. Then, the temperature in dimensionless form is ([16], p. 52)

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \int_{\tilde{u}=0}^{\tilde{t}} I_{xE}(\tilde{x},\tilde{u})\widetilde{G}_{Y21}(\tilde{y},0,\tilde{u})d\tilde{u} \tag{2}$$

where $\tilde{x} = x/L$, $\tilde{y} = y/W$, $\tilde{t} = \alpha t/W^2$, $\tilde{u} = \alpha u/W^2$ and $\widetilde{G}_{YKL} = G_{YKL}W$. In addition, we have

$$I_{xE}(\tilde{x},\tilde{u}) = \int_{\tilde{x}'=0}^{1} \sin(\beta_f\tilde{x}')\widetilde{G}_{X11}(\tilde{x},\tilde{x}',\tilde{u})d\tilde{x}' \tag{3}$$

where $\widetilde{G}_{X11} = G_{X11}L$. Eqs. (2) and (3) are now solved using the large-cotime form of Green's functions. They are given by Beck et al. ([16], p. 482, p. 491)

$$\widetilde{G}_{X11}(\tilde{x},\tilde{x}',\tilde{u}) = 2\sum_{m=1}^{\infty} e^{-C_m^2\tilde{u}}\sin(\beta_m\tilde{x})\sin(\beta_m\tilde{x}') \tag{4a}$$

$$\widetilde{G}_{Y21}(\tilde{y},\tilde{y}',\tilde{u}) = 2\sum_{n=1}^{\infty} e^{-\eta_n^2\tilde{u}}\cos(\eta_n\tilde{y})\cos(\eta_n\tilde{y}') \tag{4b}$$

where $C_m = \beta_m W/L$, $\beta_m = m\pi$ and $\eta_n = (n - 1/2)\pi$. Using Eq. (4a) in the integral (3) and re-arranging give

$$I_{xE}(\tilde{x},\tilde{u}) = 2\sum_{m=1}^{\infty} e^{-C_m^2\tilde{u}}\sin(\beta_m\tilde{x})\int_{\tilde{x}'=0}^{1}\sin(\beta_f\tilde{x}')\sin(\beta_m\tilde{x}')d\tilde{x}' \tag{5}$$

As the periodic nonhomogeneous boundary condition $X_f(\beta_f,\tilde{x}') = \sin(\beta_f\tilde{x}')$ has been chosen in such a way as to satisfy the homogeneous boundary conditions (1b) in the $x$-direction of the problem (1), we have orthogonality in Eq. (5). The result is

$$I_{xE}(\tilde{x},\tilde{u}) = \sin(\beta_f\tilde{x})e^{-C_f^2\tilde{u}} \tag{6}$$

where $C_f = \beta_f W/L$. Substituting Eq. (6) in Eq. (2) gives

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \sin(\beta_f\tilde{x})\int_{\tilde{u}=0}^{\tilde{t}} e^{-C_f^2\tilde{u}}\widetilde{G}_{Y21}(\tilde{y},0,\tilde{u})d\tilde{u} \tag{7}$$

Eq. (7) states that, for the case of spatially eigen-periodic but time invariant heat flux, the temperature solution can be written down very simply as the product of the same "eigen-periodic" surface heat flux by the solution of an integral along the nonhomogeneous direction.

Substituting Eq. (4b) in Eq. (7) and integrating over the cotime give

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \sin(\beta_f\tilde{x})\left\{2\sum_{n=1}^{\infty}\frac{\cos(\eta_n\tilde{y})}{C_f^2+\eta_n^2}\left[1-e^{-(C_f^2+\eta_n^2)\tilde{t}}\right]\right\} \tag{8}$$

## 4. Temperature solution

Eqs. (7) and (8) are important. They state that the current 2D problem "simply" reduces to an effective 1D problem in the $y$-direction. Only one single-summation appears in the solution Eq. (8) in contrast to the double summation form typical of 2D transient problems. Thus the temperature solution Eq. (8) may be rewritten as

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \sin(\beta_f\tilde{x})\theta(\tilde{y},\tilde{t},\eta_n,C_f) \tag{9}$$

where $\theta(\tilde{y},\tilde{t},\eta_n,C_f)$ is the dimensionless temperature of a 1D transient heat conduction problem along $y$. This temperature has two components, a steady-state component $\theta_{s.s.}(\tilde{y})$ and what we call the "complementary transient" component $\theta_{c.t.}(\tilde{y},\tilde{t})$. They are

$$\theta_{s.s.}(\tilde{y}) = 2\sum_{n=1}^{\infty}\frac{\cos(\eta_n\tilde{y})}{C_f^2+\eta_n^2} \tag{10a}$$

$$\theta_{c.t.}(\tilde{y},\tilde{t}) = -2\sum_{n=1}^{\infty}\frac{\cos(\eta_n\tilde{y})}{C_f^2+\eta_n^2}e^{-(C_f^2+\eta_n^2)\tilde{t}} \tag{10b}$$

### 4.1. Governing equations of the 1D problem

The governing equations of the 1D transient heat conduction problem in the $y$-direction stated before can now be derived introducing Eq. (9) into the defining Eq. (1) of the original 2D problem denoted by $X11B00$ $Y21B(xE)0T0$. In particular, introducing Eq. (9) into Eq. (1a) gives

$$\frac{\partial^2\theta}{\partial(\tilde{y})^2} - C_f^2\theta = \frac{\partial\theta}{\partial\tilde{t}} \quad (0<\tilde{y}<1,\ \tilde{t}>0) \tag{11a}$$

where $C_f = \beta_f W/L$. Introducing Eq. (9) into the remaining Eqs. (1b)–(1d) and while bearing in mind that the $X_f = \sin(\beta_f\tilde{x})$ satisfies the boundary conditions in the $x$-direction, we get a system of boundary and initial equations along $y$, that is,

$$-\left(\frac{\partial\theta}{\partial\tilde{y}}\right)_{\tilde{y}=0} = 1 \quad \theta(1,\tilde{t}) = 0 \quad \theta(\tilde{y},0) = 0 \tag{11b}$$

Eq. (11) describes a transient heat conduction problem in a fin of constant cross section aligned with the $y$-axis and having a nonhomogeneous boundary condition of the 2nd kind at $y = 0$ and a homogeneous one of the 1st kind at $y = W$. In particular, the $C_f^2\theta$ term is recognized to be the "fin" term describing the side heat losses for a fin ([16], p. 65). This fin problem may be denoted as $YF21B10T0$, where the $F$ denotes the fin.

Comparing Eqs. (7) and (9) yields the solution of the above 1D fin problem is given by the integral appearing in Eq. (7).

### 4.2. Solution procedure using "eigen-transformation"

From what has been said before, it follows that Eq. (9) may be considered as a key transformation capable of reducing the dimensions of a transient heat conduction problem when it is has a periodic-in-$x$ surface heat flux which satisfies the boundary conditions in the homogeneous direction. As in Section 2 we have termed this particular periodic surface heat flux as the "eigen-periodic" boundary heat flux, transformation (9) may hence be termed the "eigen-transformation".

Therefore, the solution of a 2D transient heat conduction problem having an "eigen-periodic" surface heat flux can be written down very simply as the product of the same "eigen-periodic" surface heat flux by the solution of a 1D fin transient problem along the nonhomogeneous direction, whose components are given by Eq. (10).

By using the algebraic identities given by Beck and Cole in Ref. ([4], Appendix B) for the *YF*21 case, the steady-state component Eq. (10a) can also be written as

$$\theta_{s.s.}(\tilde{y}) = \frac{e^{-C_f\tilde{y}} - e^{-C_f(2-\tilde{y})}}{C_f(1+e^{-2C_f})} \tag{12}$$

The above results are generalized in Appendix A where the problem denoted by *XIJB*00 *YKLB*(*xE*)0*T*0 is treated and discussed. The *I*, *J*, *K* and *L* values can be 1, 2 or 3 corresponding to the boundary condition kinds.

## 5. *X*11*B*00 *Y*2*LB*(*xE*)0*T*0 problems

Consider the two-dimensional transient heat conduction problem denoted *X*11*B*00 *Y*2*LB*(*x*6)0*T*0. This problem has a sinusoidal-in-*x* surface heat flux indicated by "(*x*6)" ([16], Chapter 2) applied to its $y = 0$ face. We can write $-k(\partial T/\partial y)_{y=0} = q_0\sin(\beta_f\tilde{x})$. This spatially periodic variation satisfies the boundary conditions in the homogeneous direction (i.e., along *x*) only if $\beta_f = f\pi$ with $f = 1, 2, 3, \ldots$ We currently assume that this is verified; hence, the sinusoidal-in-*x* surface condition becomes an "eigen-sinusoidal" condition, that is, "(*x*6)" ≡ "(*xE*)".

The above problem has zero as initial temperature and homogeneous boundary conditions of the first kind in the *x*-direction. The boundary condition at $y = W$ is also homogeneous but can be of any kind. The *L* values can in fact be 1, 2, 3, 0 or *C*0 corresponding to the boundary condition kinds ([16], Chapter 2). In particular, the boundary condition of the zero-th kind indicates that the solid is semi-infinite; the "C0" however denotes a perfect thermal contact of the first layer to a semi-infinite body (from *W* to infinity and $0 < x < L$).

The boundary condition of the first kind ($L = 1$) at $y = W$ was carefully analyzed in the previous sections. The other kinds are discussed in next subsections.

### 5.1. X11B00 Y22B(xE)0T0 problem

The solution of this problem is in the form of Eq. (9) where $C_f = f\pi(W/L)$ and $\theta$ is the solution of the 1D fin transient problem denoted by *YF*22*B*10*T*0. This solution is given by the integral appearing in Eq. (7) where $\widetilde{G}_{Y21}(\tilde{y},\tilde{y}',\tilde{u})$ has to be replaced with $\widetilde{G}_{Y22}(\tilde{y},\tilde{y}',\tilde{u})$ defined as ([16], p. 492)

$$\widetilde{G}_{Y22}(\tilde{y},\tilde{y}',\tilde{u}) = 1 + 2\sum_{n=1}^{\infty} e^{-\eta_n^2\tilde{u}}\cos(\eta_n\tilde{y})\cos(\eta_n\tilde{y}') \tag{13}$$

where $\eta_n = n\pi$. Solving the above integral gives the steady state $\theta_{s.s.}$ and complementary transient $\theta_{c.t.}$ components of $\theta$, that is,

$$\theta_{s.s.}(\tilde{y}) = \frac{1}{C_f^2} + 2\sum_{n=1}^{\infty}\frac{\cos(n\pi\tilde{y})}{C_f^2 + (n\pi)^2} \tag{14a}$$

$$\theta_{c.t.}(\tilde{y},\tilde{t}) = -\left[\frac{e^{-C_f^2\tilde{t}}}{C_f^2} + 2\sum_{n=1}^{\infty}\frac{\cos(\eta_n\tilde{y})}{C_f^2+\eta_n^2}e^{-(C_f^2+\eta_n^2)\tilde{t}}\right] \tag{14b}$$

By using the algebraic identities given in Ref. ([4], Appendix B) for the *YF*22 case, the steady-state component Eq. (14a) can also be written as

$$\theta_{s.s.}(\tilde{y}) = \frac{e^{-C_f\tilde{y}+e^{-C_f(2-\tilde{y})}}}{C_f(1-e^{-2C_f})} \tag{15}$$

The temperature solution is the sum of the above two parts multiplied by the $\sin(\beta_f\tilde{x})$ function in Eq. (9). It is explicitly given by

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \sin(\beta_f\tilde{x})\left\{\frac{e^{-C_f\tilde{y}}+e^{-C_f(2-\tilde{y})}}{C_f(1-e^{-2C_f})} - \left[\frac{e^{-C_f^2\tilde{t}}}{C_f^2} + 2\sum_{n=1}^{\infty}\frac{\cos(\eta_n\tilde{y})}{C_f^2+\eta_n^2}e^{-(C_f^2+\eta_n^2)\tilde{t}}\right]\right\} \tag{16}$$

The above summation converges rapidly with only a few terms needed except for small values of $\tilde{t}$; for example, for errors less than $10^{-5}$, five or fewer terms are needed.

### 5.2. X11B00 Y23B(xE)0T0 problem

In this case, the boundary surface at $y = W$ exchanges heat by convection with the surrounding ambient kept at zero temperature. Following the same procedure, the solution to this problem is taken as the product of the spatial eigen-periodic heat flux by the solution of the fin transient problem denoted *YF*23*B*10*T*0. It is

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \sin(\beta_f\tilde{x})\left\{\frac{e^{-C_f\tilde{y}}+D_We^{-C_f(2-\tilde{y})}}{C_f(1-D_We^{-2C_f})} - 2\sum_{n=1}^{\infty}\frac{(\eta_n^2+B_W^2)\cos(\eta_n\tilde{y})e^{-(C_f^2+\eta_n^2)\tilde{t}}}{(\eta_n^2+B_W^2+B_W)(C_f^2+\eta_n^2)}\right\} \tag{17}$$

where $C_f = \beta_f(W/L)$, $\eta_n\tan\eta_n = B_W$ and $D_W = (C_f - B_W)/(C_f + B_W)$. Also, $B_W = h_WW/k$ is the Biot number at $y = W$, where $h_W$ is the heat transfer coefficient.

Note that in Eq. (17) we have used the algebraic identities given in Ref. ([4], Appendix B) for the *YF*23 case. Also, when $B_W \to 0$, Eq. (17) reduces to Eq. (16); on the contrary, when $B_W \to \infty$, Eq. (17) reduces to Eq. (8) whose steady state part is also given by Eq. (12). This confirms that the *YF*21 and *YF*22 cases bracket the *YF*23 case. For that reason, the case of convective boundary condition ($L = 3$) is not explicitly considered afterwards.

### 5.3. X11B00 Y20B(xE)T0 problem

Let us now assume that the solid is semi-infinite in the *y*-direction. The solution to this problem may still be written in the form of Eq. (9), that is,

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \sin(\beta_f\tilde{x})\theta_{YF20}(\tilde{y},\tilde{t}) \tag{18}$$

where $\beta_f = f\pi$ with $f = 1, 2, 3, \ldots$ and $\theta_{YF20}(\tilde{y},\tilde{t})$ is the solution of the 1D "long" fin transient problem along *y*. It may be found by using the GF method. This solution is given by the integral appearing in Eq. (7) where $\widetilde{G}_{Y21}(\tilde{y},\tilde{y}',\tilde{u})$ has to be replaced with $\widetilde{G}_{Y20}(\tilde{y},\tilde{y}',\tilde{u})$ defined as ([16], p. 489)

$$\widetilde{G}_{Y20}(\tilde{y},0,\tilde{u}) = \frac{1}{\sqrt{\pi\tilde{u}}}e^{-\frac{\tilde{y}^2}{4\tilde{u}}} \tag{19}$$

Solving the resultant equation using integral # 12 of Ref. ([16], p. 428) gives $\theta_{YF20}(\tilde{y},\tilde{t})$. The temperature solution is

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{q_0W/k} = \frac{1}{2C_f}\sin(\beta_f\tilde{x})\left[-e^{C_f\tilde{y}}\operatorname{erfc}\left(\frac{\tilde{y}}{2\sqrt{\tilde{t}}}+C_f\sqrt{\tilde{t}}\right) + e^{-C_f\tilde{y}}\operatorname{erfc}\left(\frac{\tilde{y}}{2\sqrt{\tilde{t}}}-C_f\sqrt{\tilde{t}}\right)\right] \tag{20}$$

Notice that this transient 2D solution does not have any summation. For time going to infinity, we have the steady-state component

$$\frac{T(\tilde{x},\tilde{y},\infty)}{q_0W/k} = \sin(\beta_f\tilde{x})\theta_{YF20}(\tilde{y},\infty) \quad \theta_{YF20}(\tilde{y},\infty)C_f = e^{-C_f\tilde{y}} \tag{21}$$

which has a simple exponential decaying with the $y$-coordinate (typical steady behaviour of "long" fins [20]).

### 5.4. X11B00 Y2C0B(xE) problem

Consider the steady state part of the heat conduction problem where a 2D plate $L$ by $W$ with thermal conductivity $k_1$ is attached to a semi-infinite body (from $W$ to infinity and $0 < x < L$) having $k_2$ as thermal conductivity. The interface contact is considered to be perfect and the boundary conditions at $x = 0$ and $x = L$ are homogeneous and of the first kind. An "eigen-sinusoidal"-in-$x$ surface condition of the 2nd kind is applied at the $y = 0$ face of the first layer.

The above problem is denoted $X11B00\ Y2C0B(xE)$, where "$C$" would indicate a perfect thermal contact. It is mathematically described by

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = 0 \quad (0 < x < L,\ 0 < y < W) \tag{22a}$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0 \quad (0 < x < L,\ W < y < \infty) \tag{22b}$$

$$T_1(0,y) = 0, \quad T_1(L,y) = 0 \tag{22c}$$

$$T_2(0,y) = 0, \quad T_2(L,y) = 0 \tag{22d}$$

$$-k_1\left(\frac{\partial T_1}{\partial y}\right)_{y=0} = q_0 \sin\left(\beta_f \frac{x}{L}\right), \quad T_2(x,\infty) = \text{finite} \tag{22e}$$

$$T_1(x,W) = T_2(x,W) \quad k_1\left(\frac{\partial T_1}{\partial y}\right)_{y=W} = k_2\left(\frac{\partial T_2}{\partial y}\right)_{y=W} \tag{22f}$$

where $\beta_f = f\pi$ with $f = 1, 2, 3, \ldots$ Based on analyses for homogeneous bodies with "eigen-periodic"-in-space boundary heat fluxes (Sections 2–4), it is reasonable to try to obtain a solution for this problem using Eq. (9). We have

$$\frac{T_1(\tilde{x},\tilde{y})}{q_0 W/k_1} = \sin(\beta_f \tilde{x})\theta_1(\tilde{y}) \tag{23a}$$

$$\frac{T_2(\tilde{x},\tilde{y})}{q_0 W/k_1} = K\sin(\beta_f \tilde{x})\theta_2(\tilde{y}) \tag{23b}$$

where $\theta_1$ and $\theta_2$ are the solution of a 1D two-layer fin steady-state problem along the nonhomogeneous direction. Also, $K = k_2/k_1$.

The defining equations of the above fin problem denoted by $YF2C0B1$ can be derived introducing Eq. (23) into the governing Eq. (22) of the original 2D problem. As the boundary conditions in the $x$-direction are satisfied, we obtain

$$\frac{d^2\theta_1}{d\tilde{y}^2} - C_f^2\theta_1 = 0, \quad \frac{d^2\theta_2}{d\tilde{y}^2} - C_f^2\theta_2 = 0 \tag{24a}$$

$$-\left(\frac{d\theta_1}{d\tilde{y}}\right)_{\tilde{y}=0} = \frac{q_0 W}{k_1}, \quad \theta_2(\infty) = \text{finite} \tag{24b}$$

$$\theta_1(1) = \theta_2(1), \quad \left(\frac{d\theta_1}{dy}\right)_{\tilde{y}=1} = K\left(\frac{d\theta_2}{dy}\right)_{\tilde{y}=1} \tag{24c}$$

where $C_f = \beta_f W/L$. The solution is a linear combination of exponentials

$$\theta_1(\tilde{y}) = \frac{1}{C_f}\frac{e^{-C_f\tilde{y}} + \kappa e^{-C_f(2-\tilde{y})}}{1 - \kappa e^{-2C_f}} \tag{25a}$$

$$\theta_2(\tilde{y}) = \frac{1}{C_f}\frac{1}{1+K}\frac{2e^{-C_f\tilde{y}}}{1 - \kappa e^{-2C_f}} \tag{25b}$$

where $\kappa = (1 - K)/(1 + K)$. Note that for a zero conductivity second material or, in other words, a perfect insulator, $K = 0$ ($\Rightarrow \kappa = 1$), which is the same as for a 2D finite plate case insulated at $y = W$ and treated in Section 5.1 ($Y22$ case). For an infinite conductivity second material (ideal conductor), however, $K \to \infty$ ($\Rightarrow \kappa = -1$), which is the same as for a 2D finite plate case kept at zero temperature at $y = W$ and analyzed in Sections 2–4 ($Y21$ case). This confirms that the $YF21$ and $YF22$ cases bracket the $YF2C0$ case. For that reason, the two-layer case ($L = C0$) is not explicitly considered afterwards.

Also, for $K = 1$ ($\Rightarrow \kappa = 0$), Eqs. (25a) and (25b) reduce to the well-known solution of the 1D "long" fin steady-state problem (see Section 5.3).

### 5.5. Steady-state components of the X11B00 Y2LB(xE)0T0 problems

It follows from the complementary transient component of Eqs. (8) and (16) that the steady state condition may be considered reached with error less than $10^{-n}$ when

$$e^{-(C_f^2+\eta_1^2)\tilde{t}} < 10^{-n} \Rightarrow \tilde{t} > \tilde{t}_{s.s.} = \frac{n\ln(10)}{C_f^2 + \eta_1^2} \tag{26}$$

A conservative estimate of the above dimensionless time $\tilde{t}_{s.s.}$ can be obtained for $\eta_1 = \pi/2$ ($YF21$ case). Note that Eq. (26) with $\eta_1 = \pi/2$ gives a conservative estimate also for a two-layer configuration. In fact, as has already been observed before, the $YF21$ ($\eta_1 = \pi/2$) and $YF22$ ($\eta_1 = \pi$) cases bracket the $YF2C0$ case.

Consider now the steady-state component of the solutions listed before. Then this equation becomes

$$\frac{T(\tilde{x},\tilde{y},\infty)}{q_0 W/k} = \sin(\beta_f\tilde{x})\theta_{YF2L}(\tilde{y},\infty) \tag{27a}$$

where

$$\theta_{YF2L}(\tilde{y},\infty)C_f = \begin{cases} e^{-C_f\tilde{y}} & L = 0 \\ \frac{e^{-C_f\tilde{y}} - e^{-C_f(2-\tilde{y})}}{1+e^{-2C_f}} & L = 1 \\ \frac{e^{-C_f\tilde{y}} + e^{-C_f(2-\tilde{y})}}{1-e^{-2C_f}} & L = 2 \end{cases} \tag{27b}$$

Fig. 2 shows the effect of the boundary condition at $\tilde{y} = 1$ on the steady state solution $\theta^*_{YF2L}(\tilde{y},\infty) = C_f\theta_{YF2L}(\tilde{y},\infty)$ given by Eq. (27b) for different values of $\tilde{y}$ as a function of $C_f$. It may be noted that, for large values of $C_f$ (>5), the $YF21$ and $YF22$ cases (including the $YF23$ and $YF2C0$ cases) both reduce to the simpler $YF20$ one. In other words, for sufficiently large values of $C_f$, the thermal deviation effects on the steady state solution due to the homogeneous boundary condition (at $\tilde{y} = 1$) parallel to the heated surface (at $\tilde{y} = 0$) are negligible.

The above result suggests that we can define a deviation frequency as we defined a deviation time in Ref. [15]. However, according to what was done in [15] where the concept of penetration time was first introduced, it requires that a penetration frequency be defined here. They are treated in next two sections.

## 6. Penetration frequency

The penetration frequency is the frequency needed for the steady state temperature at a point $\tilde{y}$ in a 2D semi-infinite solid to be just affected (at the level of $10^{-n}$) by an "eigen-periodic"-in-space heating at the boundary surface $\tilde{y} = 0$.

The heated surface is indicated as being 'active' and it is responsible for the thermal penetration effects. These effects may be estimated as $\sigma(\tilde{y}) = \theta^*_{YF20}(\tilde{y},\infty)$, where $\theta^*_{YF20}(\tilde{y},\infty) = e^{-C_f\tilde{y}}$. Solving this equation analytically for different values of $\sigma = 10^{-n}$ for the dimensionless frequency $C_{f,pen}$ gives

$$C_{f,pen}\tilde{y} = n\ln(10) \quad \text{or} \quad \nu_{f,pen} = \frac{n\ln(10)}{y} \tag{28}$$

In fact, as $C_f = \beta_f(W/L)$, $\beta_f = \nu_f L$ and $\tilde{y} = y/W$, we can write: $C_f\tilde{y} = \nu_f y$, where $\nu_f$ is the frequency of the applied "eigen-sinusoidal"-in-space heating. In Eq. (28), $y$ is the distance between the 'active' thermal surface located at $y = 0$ and the point of interest $y$, as shown in Fig. 3a. A plot of the penetration frequency $C_{f,pen}$ versus $\tilde{y}$ for three different values of $\sigma = 10^{-n}$ is given in Fig. 4a.

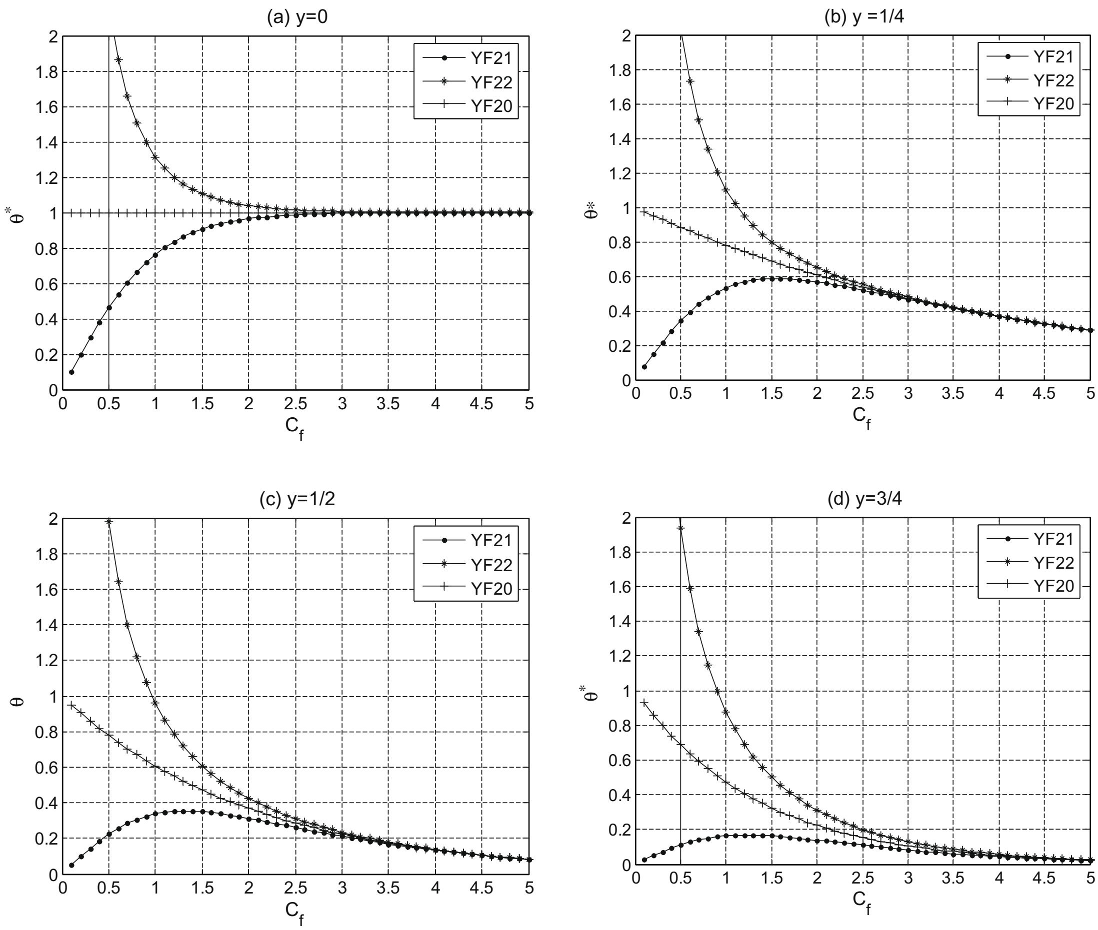


**Fig. 2.** $\theta^*_{YF2L}(\tilde{y},\infty) = C_f\,\theta_{YF2L}(\tilde{y},\infty)$ of Eq. (27b) plotted versus $C_f = \beta_f(W/L)$ for different boundary conditions at $\tilde{y} = 1$, namely $L$ = 0, 1 or 2. Solution for steady state at: (a) $\tilde{y} = 0$; (b) $\tilde{y} = 1/4$; (c) $\tilde{y} = 1/2$ and (d) $\tilde{y} = 3/4$.

The important and amazing point is that there is a relatively little difference for the dimensionless penetration frequencies when $n$ varies from 2 to 10. In fact, decreasing the penetration effects on steady state temperature from $10^{-2}$ to $10^{-10}$, a factor of $10^8$, results in an increase in the dimensionless penetration frequency only from 4.6 to 23, that is only a factor of 5.

Eq. (28) also provides a conservative estimate for the penetration distance $y_{pen}$, that is, the distance from the boundary surface $y = 0$ (heated through an "eigen-periodic"-in-space source) of a 2D semi-infinite solid at which the steady state temperature is just affected at a given frequency $\nu_f$ by this heating (Fig. 3a). Thus, we have

$$C_f\tilde{y}_{pen} = n\ln(10) \quad \text{or} \quad y_{pen} = \frac{n\ln(10)}{\nu_f} \tag{29}$$

where $y_{pen}\nu_f$ is the dimensionless penetration distance. Decreasing the penetration effects from $10^{-2}$ to $10^{-10}$ results in an increase in the penetration distance from about 4.6 to 23 (only a factor of 5), where the constant 4.6 is not far from the well-known constant 5 occurring in the laminar boundary layer thickness [20].

## 7. Deviation frequency

The deviation frequency is the frequency needed for the steady state temperature at a point $\tilde{y}$ in a 2D finite solid (heated at a boundary through an "eigen-periodic"-in-space source) to be just affected by the presence of a homogeneous boundary condition. This boundary (called 'inactive') can be parallel or perpendicular to the heated surface) and can cause thermal deviation effects. They are discussed in next two subsections.

### 7.1. Homogeneous boundary parallel to the heating surface

The deviation effects due to this boundary may be estimated as

$$\varepsilon(\tilde{y}) = |\theta^*_{YF2L}(\tilde{y},\infty) - \theta^*_{YF20}(\tilde{y},\infty)| \quad (L = 1 \text{ or } 2) \tag{30}$$

Substituting Eq. (27b) in Eq. (30) and solving analytically for different values of $\varepsilon = 10^{-n}$ for the dimensionless frequency $C_{f,dev}$ give

$$C_{f,dev} = \frac{1}{2}\ln[1 + 2\cdot 10^n\cosh(C_{f,dev}\tilde{y})] \quad (L = 1 \text{ or } 2) \tag{31}$$

For large $C_{f,dev}\tilde{y}$ we obtain $2\cosh(C_{f,dev}\tilde{y}) \approx e^{C_{f,dev}\tilde{y}}$. Then, the previous equation becomes

$$C_{f,dev}(2-\tilde{y}) \approx n\ln(10) \quad \text{or} \quad \nu_{f,dev} \approx \frac{n\ln(10)}{(2W-y)} \tag{32}$$

It is relevant to note that the above length $(2W - y)$ is greater than the distance between the 'inactive' surface (homogeneous boundary condition) at $y = W$ and the point of interest $y$, namely $(W - y)$, as shown in Fig. 3b. This result is in accordance with what was found for the deviation time due to an 'inactive' boundary parallel to an 'active' one in transient heat conduction (see Section 4 of Ref. [15]). A plot of the deviation frequency $C_{f,dev}$ versus $\tilde{y}$ for three different values of $\varepsilon = 10^{-n}$ is given in Fig. 4b.

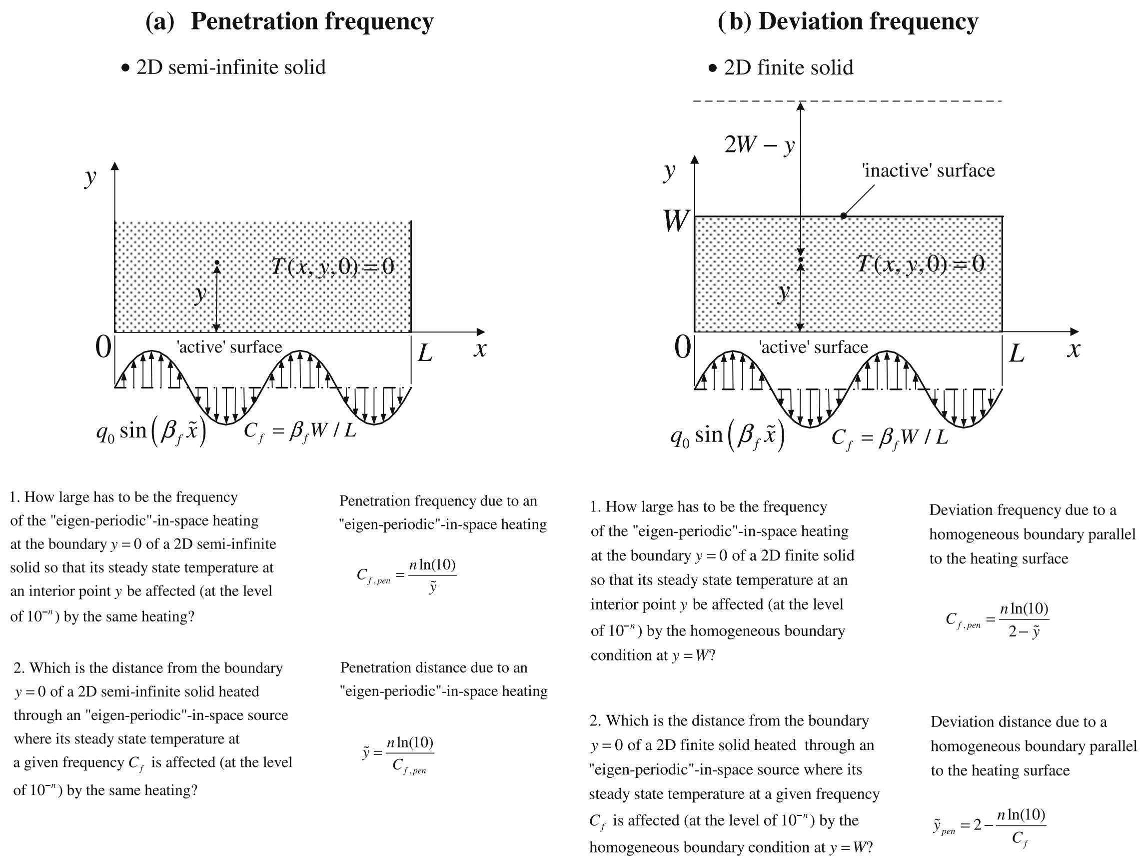


**Fig. 3.** "Eigen-periodic"-in-space heating at the boundary $y = 0$.

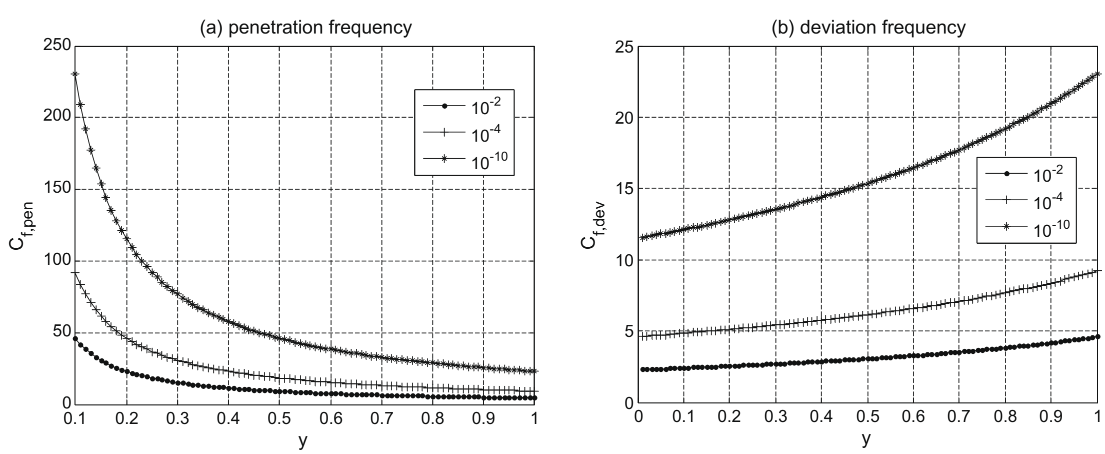


**Fig. 4.** Characteristic frequencies $C_f$ as a function of $\tilde{y}$ for different values of $10^{-n}$. (a) penetration frequency due to an "eigen-periodic"-in-space heating at the boundary $y = 0$, and (b) deviation frequency due to the homogeneous boundary $y = W$ parallel to the 'active' surface $y = 0$ heated through an "eigen-periodic"-in-space source.

Eq. (32) also provides a conservative estimate for the deviation distance $y_{dev}$, that is, the distance measured from the boundary $y = 0$ (heated through an "eigen-periodic"-in-space source) of a 2D finite solid at which the steady state temperature is just affected at a given frequency $\nu_f$ by the homogeneous boundary condition at $y = W$ (Fig. 3b). Thus, we have

$$C_f(2 - \tilde{y}_{dev}) \approx n\ln(10) \quad \text{or} \quad y_{dev} \approx 2W - \frac{n\ln(10)}{\nu_f} \tag{33}$$

where $y_{dev}\nu_f$ is the dimensionless deviation distance.

As regards the boundary at $y = W$, also a second material could begin there (followed by as many layers as desired; the contact could be perfect or imperfect); or some nonlinear condition such as a radiation boundary condition which would again be "homogeneous," that is, not introducing heating or cooling at that surface until the effect of the $y = 0$ boundary reaches there. At $y = W$ there could even be freezing or melting.

For all of the above possible conditions at $y = W$, Eqs. (32) and (33) are still valid. For instance, if we consider the two-layer steady state eigen-periodic problem of Section 5.4, the thermal deviation effects due to the semi-infinite second layer may be estimated as

$$\varepsilon(\tilde{y}) = |\theta^*_{1,YF2C0}(\tilde{y},\infty) - \theta^*_{YF20}(\tilde{y},\infty)| \tag{34}$$

where $\theta^*_{1,YF2C0}(\tilde{y},\infty) = C_f\theta_{1,YF2C0}(\tilde{y},\infty)$ is given by Eq. (25a). Substituting this equation in Eq. (34) and solving for different values of $\varepsilon = 10^{-n}$ for the dimensionless frequency $C_{f,dev}$ give

$$C_{f,dev} = \frac{1}{2}\ln[1 + 2\cdot 10^n \cosh(C_{f,dev}\tilde{y})] + \ln|\kappa| \tag{35}$$

It differs from Eq. (31) because of the term related to $\kappa$. For large $C_{f,dev}\tilde{y}$ it becomes

$$C_{f,dev}(2-\tilde{y}) \approx n\ln(10) + 2\ln|\kappa| \quad (|\kappa| \in (0,1]) \tag{36}$$

The values possible for $\kappa$ are between −1 and 1. For $k_2 > k_1$, we have $\kappa \in [-1, 0)$. On the contrary, for $k_2 < k_1$, we have $\kappa \in (0, 1]$. For the special case of $\kappa = 0$ (i.e., $k_2 = k_1$), the two-layer configuration reduces to a semi-infinite body and, hence, there exist no thermal deviation effects on the steady state temperature. For $\kappa = \pm 1$, Eq. (36) reduces to the simpler Eq. (32). It is also interesting to observe that the deviation frequencies given by Eq. (36) are smaller than the corresponding ones obtainable for $|\kappa| = 1$ because ln $|\kappa|$ is always less than zero. Therefore, Eq. (36) can also be used in this case as a conservative estimate for the deviation frequencies.

### 7.2. Homogeneous boundary perpendicular to the heating surface

When the frequency $\beta_f$ of the applied thermal force $q_0\sin(\beta_f\tilde{x})$ equates one of the natural frequencies (or eigenvalues) of the system, the homogeneous boundary conditions at $x = 0$ and $x = L$ (perpendicular to the heating surface) cause no thermal deviation effects on the steady state temperature. In such a case, in fact, the temperature is given by Eqs. (27) and the problem is reduced from 2D to 1D.

## 8. Thermal conductivity in thin films

From what was said in the previous sections, one solution for determining the thermal conductivity of a thin film placed on a substrate can be based on a time-independent, spatial sinusoidal-variation surface heat flux, as shown in Fig. 5. This sinusoidal-in-space heating may experimentally be obtained by using the pulsed-laser interference fringes [7–9].

A pulsed high-power laser beam can be divided by a beam splitter into two beams of equal intensity. These two beams of spatial and longitudinal coherence can be intersected on the thin layer surface $y = 0$ by paraboloidal mirrors under an angle $\vartheta$ and generate an optical interference fringe pattern whose intensity distribution is spatially sinusoidal. Its frequency $\nu_f$ is the so-called wave number of the fringe given by $2\pi/\Lambda$, where $\Lambda$ is the grating period or fringe spacing. This period may be taken as $\Lambda = \lambda/[2\sin(2\vartheta)]$, where $\lambda$ is the laser wavelength and $\vartheta$ is the interference angle in radians. Therefore, by simply changing the crossing angle $\vartheta$ of the two pump beams, the above period can adequately be varied.

However, in the thermal grating technique [7–9] (which is a time-domain technique) used for thin films, the pulsed-laser excitation takes a short while so allowing measurements of the thermal diffusivity. In contrast, in proposed technique the pulsed-laser excitation is applied until the steady state condition is reached. As a matter of fact, the solution does have a steady state but the implementation experimentally has a quasi-steady state. We need to add a constant heat flux component so that the heat flux never becomes negative, as shown afterwards.

As we have a thin layer-substrate composite, the steady state surface temperature at $y = 0$ can be obtained by using the simple Eqs. (23a) and (25a) provided the periodic-in-space heating be eigen-sinusoidal, that is, $\nu_f \equiv f\pi/L$ ($f = 1, 2, 3, \ldots$), where $\nu_f = 4\pi \sin(2\vartheta)/\lambda$. Therefore, we have

$$\frac{T_1(\tilde{x},0)}{q_0W/k_1} = \sin(\beta_f\tilde{x})\frac{1}{C_f}\frac{1+\kappa e^{-2C_f}}{1-\kappa e^{-2C_f}} \quad \text{if} \quad \lambda = \frac{4\sin(2\vartheta)L}{f} \tag{37}$$

The maximum temperature difference at $y = 0$ is between the points $A$ and $B$ of Fig. 5

$$\frac{T_1\left(\frac{\pi}{2\beta_f},0\right) - T_1\left(\frac{3\pi}{2\beta_f},0\right)}{q_0W/k_1} = \frac{\Delta T_{1,AB}(0)}{q_0W/k_1} = \frac{2}{C_f}\left(\frac{1+\kappa e^{-2C_f}}{1-\kappa e^{-2C_f}}\right) \tag{38}$$

As the spatial sinusoidal input is both positive and negative, it is evident that we cannot provide this condition experimentally. However, we can have a uniform heat flux of $q_m \geqslant q_0$ superimposed upon the sinusoidal-in-space heat flux, as shown in Fig. 5. The net result is simply the superposition of a sinusoidal heat flux and a positive uniform heat flux. The resulting heat flux is always positive which can be obtained experimentally.

Then, the steady state temperatures at the points $A$ and $B$ have also to account for the quasi-steady state contribution due to $q_m$. However, if the surface two points are symmetrical with respect to the middle plane $x = L/2$ of the thin film-substrate composite (such as the points $A$ and $C$ of Fig. 5) and the boundary conditions at $x = 0$ and $x = L$ are of the same kind, the temperature rise due to $q_m$ at $A$ and $C$ is exactly the same. The temperature difference is hence given by Eq. (38), where $\Delta T_{1,AB}(0)$ is now replaced with $\Delta T_{1,AC}(0)$.

Eq. (38) suggests a method to determine the thermal conductivity $k_1$ from temperature measurements at $A$ and $C$. A couple of points are needed. The first is that the $\kappa$ ratio also contains $k_1$ so Eq. (38) cannot reduce to an explicit equation for $k_1$. However, if (see Section 7.1)

$$\beta_f > \beta_{f,dev} \approx \frac{n\ln(10)}{2W/L} \Rightarrow \lambda < \frac{8\pi W\sin(2\vartheta)}{n\ln(10)} \tag{39}$$

the thermal deviation effects due to the substrate on the steady state temperature of the thin film at its boundary surface $y = 0$ are negligible at the level of one part in $10^n$. This indicates that the quantity between brackets on the RHS of Eq. (38) reduces to the unity with an error less than $10^{-n}$. Then, solving this equation for the conductivity gives

$$k_1 = \frac{1}{\beta_f}\frac{2q_0L}{\Delta T_{1,AC}(0)} \tag{40}$$

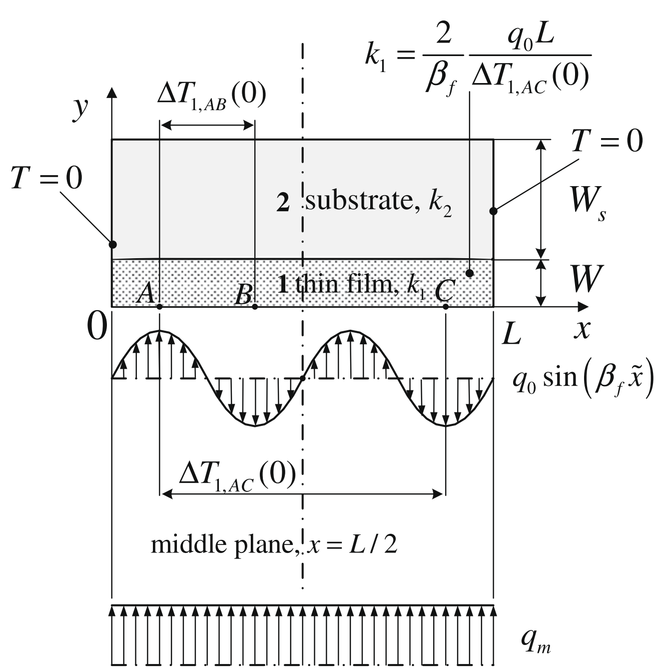


**Fig. 5.** Schematic illustrating the film-substrate geometry used in the analytical and calculations. The spatial sinusoidal heat flux $q_0\sin(\beta_f\tilde{x})$ and the positive uniform heat flux $q_m \geqslant q_0$ at $y = 0$ are indicated.

For example, if $W = 0.1$ mm, $\vartheta = 45°$ and $n = 3$ (i.e., error less than 0.1%), we have from Eq. (39) that $\lambda < 364$ μm. Therefore, it follows from constraint Eq. (37) that the integer $f$ has to be larger than $4L/(364\ \mu m)$. If $L = 10$ mm, we find that $f \geqslant 110$. Then, if we choose $f = 125$, we obtain a laser wavelength of $\lambda = 320$ μm; also, $\Lambda = 160$ μm, $\nu_f = 0.0125\pi\ \mu m^{-1}$ and $\beta_f = 125\pi$.

The second point is that, as the film becomes thin, the sinusoidal-in-space variation requires a higher frequency $\beta_f$ to accurately neglect the deviation effects due to the substrate, as indicated by Eq. (39). Consequently, as we go up in frequency, the temperature difference from peak to valley goes down, as indicated by Eq. (40).

### 8.1. Orthotropic thermal conductivity in thin films

If the thin film is orthotropic, then the heat diffusion Eq. (22a) under the steady state condition becomes

$$k_{1x}\frac{\partial^2 T_1}{\partial x^2} + k_{1y}\frac{\partial^2 T_1}{\partial y^2} = 0 \quad (0 < x < L,\ 0 < y < W) \tag{41}$$

In such a case, if Eq. (39) is satisfied, then the thermal deviation effects due to the substrate on the steady state temperature of the thin film at its boundary surface $y = 0$ are negligible and we can derive the following result

$$\sqrt{k_{1x}k_{1y}} = \frac{1}{\beta_f}\frac{2q_0 L}{\Delta T_{1,AC}(0)} \tag{42}$$

For $q_0 = 10\ W/mm^2$, $\Delta T_{1,AC}(0) = 10$ K, $\beta_f = 125\pi$ and $L = 10$ mm, we have from Eq. (42) that $\sqrt{k_{1x}k_{1y}} \approx 50$ W/(mK). If we now consider a graphite sheet [9] which has a high thermal conductivity and anisotropy in the directions $x$ and $y$ ($k_{1y} = 5$ W/(mK)), we obtain that the lateral conductivity is of $k_{1x} = 500$ W/(mK).

Finally, as the solution we propose to measure the thin film thermal conductivity is a quasi-steady state technique, it is important to know the time that it takes to reach the quasi-steady state condition. This time may be evaluated by Eq. (26) setting $\eta_1 = \pi/2$. Then, if $n = 3$, $L = 10$ mm, $W = 0.1$ mm and $\beta_f = 125\pi$ (as above), we have $C_f = 1.25\pi$ and, hence, $\tilde{t}_{s.s.} \approx 0.39$. As the thermal diffusivity of the graphite sheet is of about $600\ mm^2/s$, the time is of 6.5 μs, which is an extremely short time.

## 9. Conclusions

Transient heat conduction solutions involving periodic-in-space boundary conditions have been investigated in the paper. Special attention has been given to those boundary conditions which incorporate an eigenfunction parallel to the nonhomogeneous surface. An advantage of these solutions is that the 2D solutions reduce to 1D solutions and the 3D solutions simplify to 2D or even 1D solutions; in each the solutions are simply multiplied by the given periodic boundary eigenfunction. Although these solutions are relatively simple, we were unable to find them documented in the literature.

Several more advantages of these solutions have been discussed in the paper. The steady state solutions for finite bodies in the direction normal to the eigen-periodic surface reduce for high frequencies to those for semi-infinite bodies in that direction. These solutions have also introduced new concepts of penetration and deviation frequencies (or distances). These concepts are important. As an example, for a two-layer configuration the thermal deviation effects due to the second layer on the first layer surface temperature can be neglected (at the level of $10^{-n}$) provided the frequency $C_f$ of the spatial eigen-periodic condition be larger than the deviation frequency $C_{f,dev} \approx n\ \ln(10)/2$.

Another advantage of the solutions is the insight that they provide. Simpler solutions suggest experiments for finding thermal properties, thermal conductivity in the present case. Experiments have been performed for periodic-in-space boundary conditions so such conditions can be obtained particularly for thin layers. Finding the thermal conductivity of thin layers is very important in the electronics industry and fabrication of diamond and other films. In addition the time constant to reach a steady (or quasi-) steady state solution is very small. This can provide for rapid measurement of conductivity even during a manufacturing operation.

Finally, if the conductivity of a thin film can be measured, the same technique can be used for a larger thickness of material, provided the conductivity of the near-surface material is the same as the bulk conductivity. One can visualize a portable devise that you simply apply against a surface for a short while and get a measure of the thermal conductivity.

## Appendix A

The solution of the general problem denoted by *XIJB*00 *YKLB* (*xE*)0*T*0 may be derived using Green's functions [16,18,19]. The temperature in dimensionless form is

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{T_K} = \int_{\tilde{u}=0}^{\tilde{t}} I_{xE}(\tilde{x},\tilde{u})\left[\frac{\partial^p}{\partial(\tilde{y}')^p}\tilde{G}_{YKL}(\tilde{y},\tilde{y}',\tilde{u})\right]_{\tilde{y}'=0} d\tilde{u} \tag{A.1}$$

where the order of derivative $p = 1$ for $K = 1$ (BC of the 1st kind, $T_1 = T_0$); $p = 0$ for $K = 2$ (BC of the 2nd kind, $T_2 = q_0 W/k$); and $p = 0$ for $K = 3$ (BC of the 3rd kind, $T_3 = h_{y=0}T_\infty W/k$). In addition, we have

$$I_{xE}(\tilde{x},\tilde{u}) = \int_{\tilde{x}'=0}^{1} X_f(\beta_f,\tilde{x}')\tilde{G}_{XIJ}(\tilde{x},\tilde{x}',u^+)d\tilde{x}' \tag{A.2}$$

where $X_f(\beta_f,\tilde{x})$ is the "eigen-periodic" variation along $x$ of the nonhomogeneous boundary condition applied at $y = 0$. It is the solution of the eigenvalue problem in the $x$-direction. (See ([16], p. 99) for a complete list where the eigenvalues $\beta_f$, $f = 1, 2, \ldots$, are shown as positive roots of the appropriate eigenconditions.)

Eq. (A.1) is now solved using the large-cotime Green's functions ([16], Appendix X). They are (in dimensionless form, $\tilde{G}_{XIJ} = LG_{XIJ}$ and $\tilde{G}_{YKL} = WG_{YKL}$)

$$\tilde{G}_{XIJ}(\tilde{x},\tilde{x}',\tilde{u}) = \sum_{m=0}^{\infty}\frac{e^{-C_m^2\tilde{u}}}{\tilde{N}_{x,m}}X_m(\beta_m,\tilde{x})X_m(\beta_m,\tilde{x}') \tag{A.3a}$$

$$\tilde{G}_{YKL}(\tilde{y},\tilde{y}',\tilde{u}) = \sum_{n=0}^{\infty}\frac{e^{-\eta_n^2\tilde{u}}}{\tilde{N}_{y,n}}Y_n(\eta_n,\tilde{y})Y_n(\eta_n,\tilde{y}') \tag{A.3b}$$

where the eigenfunctions $X_m$ and $Y_n$, norms $\tilde{N}_{x,m} = N_{x,m}/L$ and $\tilde{N}_{y,n} = N_{y,n}/W$, and eigenvalues $\beta_m$ and $\eta_n$ along $x$ and $y$, respectively, are given in ([16], p. 99). Also, $C_m = \beta_m W/L$. Using Eq. (A.3a) in the integral (A.2) and re-arranging give

$$I_{xE}(\tilde{x},\tilde{u}) = \sum_{m=0}^{\infty}\frac{e^{-C_m^2\tilde{u}}}{\tilde{N}_{x,m}}X_m(\beta_m,\tilde{x})\int_{x'^+=0}^{1} X_f(\beta_f,\tilde{x}')X_m(\beta_m,\tilde{x}')d\tilde{x}' \tag{A.4}$$

As the periodic nonhomogeneous boundary condition $X_f(\beta_f,\tilde{x}')$ has been chosen in such a way as to satisfy the homogeneous boundary conditions in the $x$-direction, we have orthogonality in Eq. (A.4). The result is

$$I_{xE}(\tilde{x},\tilde{u}) = X_f(\beta_f,\tilde{x})e^{-C_f^2\tilde{u}} \tag{A.5}$$

where $C_f = \beta_f W/L$. Substituting Eq. (A.5) in Eq. (A.1) gives

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{T_K} = X_f(\beta_f,\tilde{x})\int_{\tilde{u}=0}^{\tilde{t}} e^{-C_f^2\tilde{u}}\left[\frac{\partial^p}{\partial(\tilde{y}')^p}\tilde{G}_{YKL}(\tilde{y},\tilde{y}',\tilde{u})\right]_{\tilde{y}'=0} d\tilde{u} \tag{A.6}$$

Eq. (A.6) states that, for the case of spatially eigen-periodic but time invariant surface heating, the temperature solution can be written down very simply as the product of the same "eigen-periodic" boundary condition by the solution of a 1D transient heat conduction problem along the nonhomogeneous direction $y$. Therefore, the current 2D problem "simply" reduces to an effective 1D problem in the $y$-direction.

Following the approach given in Section 4.1, it may be proven that the above problem is the transient heat conduction problem in a fin of constant cross section aligned with the $y$-axis and having a nonhomogeneous boundary condition of the $K$th kind at $y = 0$ and a homogeneous one of the $L$th kind at $y = W$. This fin problem may be denoted as *YFKLB*10*T*0.

Substituting Eq. (A.3b) in Eq. (A.6) and integrating over the cotime give

$$\frac{T(\tilde{x},\tilde{y},\tilde{t})}{T_K} = X_f(\beta_f,\tilde{x})\sum_{n=0}^{\infty}\left\{\frac{Y_n(\eta_n,\tilde{y})}{\tilde{N}_{y,n}(C_f^2+\eta_n^2)}\left[\frac{d^pY_n}{d(\tilde{y}')^p}\right]_{\tilde{y}'=0}\left[1-e^{-(C_f^2+\eta_n^2)\tilde{t}}\right]\right\} \tag{A.7}$$

Only one single-summation appears in the solution Eq. (A.7) in contrast to the double summation form typical of 2D transient problems. Then, by using the algebraic identities given by Beck and Cole in Ref. ([4], Appendix B) for the nine possible *YFKL* cases ($K$, $L$ = 1, 2, 3), the slow convergence of the steady state component in Eq. (A.7) can be avoided.

## References


[1] J.V. Beck, A. Haji-Sheikh, D.E. Amos, D.H.Y. Yen, Verification solution for partial heating of rectangular solids, Int. J. Heat Mass Transfer 47 (2004) 4243–4255.
[2] F. de Monte, Transverse eigenproblem of steady-state heat conduction for multi-dimensional two-layered slabs with automatic computation of eigenvalues, Int. J. Heat Mass Transfer 47 (2004) 191–201.
[3] F. de Monte, Multi-layer transient heat conduction using transition time scales, Int. J. Thermal Sci. 45 (2006) 882–892.
[4] J.V. Beck, K.D. Cole, Improving convergence of summations in heat conduction, Int. J. Heat Mass Transfer 50 (2007) 257–268.
[5] J.V. Beck, R. McMasters, K.J. Dowding, D.E. Amos, Intrinsic verification methods in linear heat conduction, Int. J. Heat Mass Transfer 49 (2006) 2984–2994.
[6] S.L. Shindé, J.S. Goela, High Thermal Conductivity Materials, Springer, New York, 2006.
[7] O.W. Kaeding, E. Matthias, R. Zachai, H.-J. Fußer, P. Munzinger, Thermal diffusivities of thin diamond films on silicon, Diamond Relat. Mater. 2 (1993) 185–1190.
[8] J.E. Graebner, Measurement of thermal diffusivity by optical excitation and infrared detection of a transient thermal grating, Rev. Sci. Instrum. 66 (1995) 3903–3906.
[9] Y. Taguchi, Y. Nagasaka, Thermal diffusivity measurement of high-conductivity materials by dynamic grating radiometry, International. J. Thermophys. 22 (2001) 289–299.
[10] H. Abe, T. Kanno, M. Kawai, K. Suzuki, Engineering Ceramics, Gihodo-Shuppan, Tokyo, 1984.
[11] K. Watari, High thermal conductivity non-oxide ceramics, J. Ceramic Soc. Jpn. 109 (2001) S7–S16.
[12] D. Poulikakos, A. Bejan, Penetrative convection in porous medium bounded by a horizontal wall with hot and cold spots, Int. J. Heat Mass Transfer 27 (10) (1984) 1749–1757.
[13] R. Bradean, D.B. Ingham, P.J. Heggs, I. Pop, Buoyancyinduced flow adjacent to a periodically heated and cooled horizontal surface in porous media, Int. J. Heat Mass Transfer 39 (3) (1996) 615–630.
[14] J.-S. Yoo, Thermal convection in a vertical porous slot with spatially periodic boundary temperatures: low Ra flow, Int. J. Heat Mass Transfer 46 (2) (2003) 381–384.
[15] F. de Monte, J.V. Beck, D.E. Amos, Diffusion of thermal disturbances in two-dimensional Cartesian transient heat conduction, Int. J. Heat Mass Transfer 51 (25–26) (2008) 5931–5941.
[16] J.V. Beck, K.D. Cole, A. Haji-Sheikh, B. Litkouhi, Heat Conduction Using Green's Functions, Hemisphere Press, Washington, 1992.
[17] M.N. Ozisik, Heat Conduction, second ed., John Wiley, 1993.
[18] Z.H. Wang, S.K. Au, K.H. Tan, Heat transfer analysis using a Green's function approach for uniformly insulated steel members subject to fire, Eng. Struct. 27 (10) (2005) 1551–1562.
[19] Z.H. Wang, K.H. Tan, Green's function approach for heat conduction: application to steel members protected by intumescent paint, Numer. Heat Transfer B: Fundam. 54 (6) (2008) 435–453.
[20] F.P. Incropera, D.P. DeWitt, Introduction to Heat Transfer, third ed., John Wiley, 1996.